Quaternion Controller Design Using Switched Linear Parameter-Varying Framework

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I. Introduction

SPACECRAFT are often required to perform large angle eigenaxis maneuvers for various missions such as spacecraft formation flying. In this case, the spacecraft dynamics are highly nonlinear. Such maneuvers have been achieved by feedforward and feedback control approaches.[1] In particular, the feedback control approach is often based on the classic Lyapunov stability theorem except for a recent work based on quaternion factorization.[2]

This Note proposes a new technique of designing quaternion feedback controllers to facilitate fuel-efficient eigenaxis rotations with a switched linear parameter-varying (SLPV) dissipation framework.[3] The spacecraft dynamics are first formulated into an SLPV system. A stabilizing feedback controller for this SLPV spacecraft model is then derived by the SLPV dissipation approach. The designed controller is shown to exhibit more fuel-efficient eigenaxis rotations than the previously published approach.[1] This improvement is due to the result that the designed controller can utilize the cross-coupling terms of the quaternion, which are often ignored due to limitations of available design techniques.

II. Spacecraft Dynamics

Some notations are established in this Note. Let $\boldsymbol{\omega}$ be the angular velocity, J be the inertia matrix, $\boldsymbol{u}$ be the control torque, and $Q_{k=1,\ldots,4}$ be the quaternion elements.[1] Let I be the identity matrix and an interval product function be

$$\prod_{k=1}^{n}[c_{k1}, c_{k2}] = [c_{11}, c_{12}] \times \cdots \times [c_{n1}, c_{n2}]$$

Here, $\tilde{\omega}$ is defined as a skew-symmetric matrix of a vector $\boldsymbol{\omega}$ to represent the vector cross operator. In particular, $\mathrm{sign}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} = 0$.

Consider spacecraft under large angle maneuvers. The spacecraft dynamics can be described as the Euler equation and the quaternion kinematic equation.[1] However, the typical quaternion has the potential numerical issue in that the quaternion elements have different magnitudes near the equilibrium condition, that is, $Q_{k=1,2,3}(t) \to 0$ and $Q_4(t) \to \pm 1$. Thus, the original quaternion is replaced by an error quaternion: $q_k = Q_k$ and $q_4 = \mathrm{sign}(Q_4) - Q_4$. In this case, the relationship between error quaternion and angular velocity is given as follows. With $\boldsymbol{q} = [q_1 \;\; q_2 \;\; q_3]^T$,

$$\dot{\boldsymbol{q}} = \tfrac{1}{2}\tilde{q}\boldsymbol{\omega} + \tfrac{1}{2}[\mathrm{sign}(q_4) - q_4]\boldsymbol{\omega}, \qquad \dot{q}_4 = \tfrac{1}{2}\boldsymbol{q}^T\boldsymbol{\omega} \tag{1}$$

Note that if $Q_4(t)$ never crosses the zero line, these equations are exactly equivalent to the quaternion equations of Ref. 1. Otherwise,

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these equations have unique locally Lipschitz trajectories in t because the discontinuity of $\mathrm{sign}(q_4)$ occurs according to a well-defined smooth signal $Q_4(\cdot)$ (see Ref. 3). Also, note that the error quaternion satisfies the constraint that its magnitude is unit.

As a result, the spacecraft dynamics under large-angle maneuvers can be described by the Euler equation and the Lipschitz trajectories of the error quaternion equations:

$$\dot{\boldsymbol{x}} = A(\boldsymbol{x})\boldsymbol{x} + B_u\boldsymbol{u}$$

$$= \begin{bmatrix} -J^{-1}\tilde{\omega}J & 0 & 0 \\ \frac{1}{2}[\mathrm{sign}(q_4) - q_4]I + \frac{1}{2}\tilde{q} & 0 & 0 \\ \frac{1}{2}\boldsymbol{q}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{q} \\ q_4 \end{bmatrix} + \begin{bmatrix} J^{-1} \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u} \quad (2)$$

Note that this nonlinear equation is the best system representation that can facilitate the design of general quaternion controllers within the SLPV dissipation approach. In what follows, it is assumed that the angular velocity and the error quaternion are measurable.

III. Quaternion Controller Design

We design a stabilizing controller with the SLPV dissipation approach.[3] First, Eq. (2) is conservatively modeled into a quasi-SLPV system with three assumptions: 1) the $\boldsymbol{x}$ of $A(\boldsymbol{x})$ is the parameter $\boldsymbol{\theta}$ that is independent of the state x, 2) the unit-norm quaternion constraint is ignored (which implies $-1 \le q_{k=1,\dots,4} \le 1$), and 3) the angular velocity $\boldsymbol{\omega}$ is bounded, that is, $C_{k1} \le \omega_k \le C_{k2}$, $k = 1, 2, 3$. Note that these bounds must be carefully selected to be larger than the expected maximum variation of $\boldsymbol{\omega}$. With these assumptions, we can define the parameter space

$$\mathcal{F} = \prod_{k=1}^{3}[C_{k1}, C_{k2}] \times \prod_{j=1}^{4}[-1, 1]$$

Obviously, this space is the union of two half subspaces, $\mathcal{F}_1$ and $\mathcal{F}_2$, that are the subspaces of $q_4 \ge 0$ and $q_4 \le 0$, respectively. As a result, Eq. (2) can be formulated as an SLPV system

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} \alpha_i(\boldsymbol{\theta})[A_i(\boldsymbol{\theta})\boldsymbol{x} + B_u\boldsymbol{u}] \quad (3)$$

where $\alpha_i(\boldsymbol{\theta}) = 1$ if $\boldsymbol{\theta} \in \mathcal{F}_i$ and otherwise $\alpha_i(\boldsymbol{\theta}) = 0$. Note that the parameter $\boldsymbol{\theta}$ is generally piecewise smooth and, thus, is different from the counterpart of SLPV systems.[3] However, Lim and Chan[4] recently showed that this difference has no effect on the SLPV analysis and synthesis. Also, note that each $A_i(\boldsymbol{\theta})$ is an affine matrix function of $\boldsymbol{\theta}$ (Ref. 5).

We consider a full-state SLPV feedback controller:

$$\boldsymbol{u} = -\sum_{i=1}^{2} \alpha_i(\boldsymbol{\theta})K_i(\boldsymbol{\theta})\boldsymbol{x}$$

and each $K_i(\boldsymbol{\theta})$ is continuous in $\boldsymbol{\theta}$ over $\mathcal{F}_i$. These $K_i(\boldsymbol{\theta})$ can be found by the SLPV dissipation approach with quadratic storage function.[3] First, we derive a stability condition of the closed-loop SLPV system: The locally Lipschitz trajectories of the closed-loop SLPV system are uniformly stable if there exists a positive matrix P such that for $i = 1, 2$,

$$[A_i(\boldsymbol{\theta}) - B_uK_i(\boldsymbol{\theta})]^TP + P[A_i(\boldsymbol{\theta}) - B_uK_i(\boldsymbol{\theta})] \le 0, \quad \forall \boldsymbol{\theta} \in \mathcal{F}_i \quad (4)$$

This equation can be transformed into a more computationally tractable formulation: With $S = P^{-1}$ and $Y_i(\boldsymbol{\theta}) = K_i(\boldsymbol{\theta})S$,

$$A_i(\boldsymbol{\theta})S + SA_i^T(\boldsymbol{\theta}) - B_uY_i(\boldsymbol{\theta}) - Y_i(\boldsymbol{\theta})^TB_u^T \le 0, \quad \forall \boldsymbol{\theta} \in \mathcal{F}_i \quad (5)$$

In particular, $Y_i(\boldsymbol{\theta})$ is assumed to an affine function of $\boldsymbol{\theta}$. In accord with convexity, Eq. (5) can be further reduced to the following:

$$A_i(\boldsymbol{\nu})S + SA_i^T(\boldsymbol{\nu}) - B_uY_i(\boldsymbol{\nu}) - Y_i(\boldsymbol{\nu})^TB_u^T \le 0, \quad \forall \boldsymbol{\nu} \in \mathcal{C}(\mathcal{F}_i) \quad (6)$$

where $\mathcal{C}(\mathcal{F}_i)$ is the set of the vertices of $\mathcal{F}_i$. When this inequality is solved, the $K_i(\boldsymbol{\theta})$ can be derived. Note that because the parameter $\boldsymbol{\theta}$ is actually $\boldsymbol{x}$, the SLPV controller means

$$-\sum_{i=1}^{2} \alpha_i(\boldsymbol{x})K_i(\boldsymbol{x})\boldsymbol{x}$$

Also, note that, although assumption 2 contributes to Eq. (6) and then reduces the computation for synthesis, it requires a postanalysis to ensure the uniform asymptotic stability because Eq. (3) contains uncontrollable and neutrally stable states for some $\boldsymbol{\theta}$, for example, $\boldsymbol{q} = 0$ and $q_4 = (-1)^i$.

A postanalysis using the well-known LaSalle's theorem is suggested. Of course, the postanalysis is performed over the original spacecraft dynamics and not the SLPV model. This theorem leads to the following: The uniform asymptotic stability is guaranteed when

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{x}^T\left[A_i(\boldsymbol{x})S + SA_i^T(\boldsymbol{x}) - B_uY_i(\boldsymbol{x}) - Y_i^T(\boldsymbol{x})B_u^T\right]\boldsymbol{x} < 0 \quad \forall \boldsymbol{x} \in \mathcal{G}_i \quad (7)$$

where

$$\mathcal{G}_i = \left\{ \boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{F}_i, \quad \sum_{k=4}^{6} x_k^2 + [\mathrm{sign}(x_7) - x_7]^2 = 1 \right\}$$

The postanalysis evaluates Eq. (7) with a heuristic method to find local maxima of $\dot{V}(\boldsymbol{x})$ from selected grids.

IV. Numerical Examples

We consider a benchmark spacecraft model with $J = \mathrm{diag}\,[10{,}000, 9000, 12{,}000]$ and maximum angular velocity limited to 0.5 rad/s. A SLPV quaternion controller is found by solving Eq. (6) with an available solver[6] and verified to exhibit the uniformly asymptotic stability.[5] For a comparison, we designed a typical linear quaternion controller with the gyroscopic computed torque.[1] In this case, the controller gain is selected so that both controllers exhibit approximately the same settling time. As shown in Ref. 5, the difference exists in that the SLPV controller contains new quaternion cross-coupling terms q_iq_4 for ith axis.

Both controllers are used to close the feedback loop of the benchmark spacecraft. The controlled spacecraft are simulated with the same initial condition. In particular, each feedback controller is turned on at 100 s. Figures 1–3 exhibit the responses of the angular velocity, control torque, and quaternion. Figure 1 shows that the SLPV controller requires less kinetic energy

$$\frac{1}{2}\int \boldsymbol{\omega}^TJ\boldsymbol{\omega}\,\mathrm{d}t$$

during maneuvering than the linear controller. This result is also confirmed by Fig. 2, which shows that the SLPV controller reduces

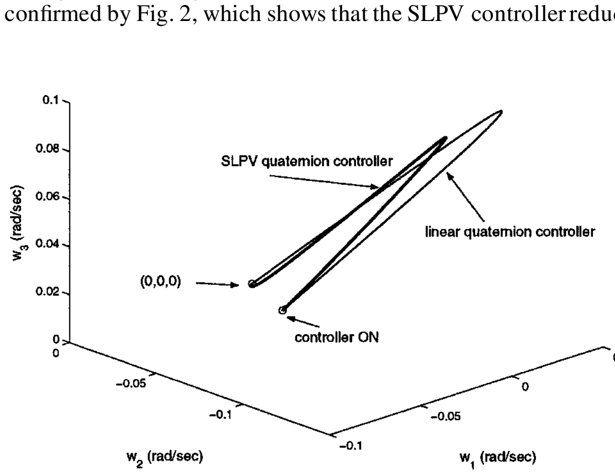

Fig. 1 Response of angular velocity ω_k, $k = 1, 2, 3$.

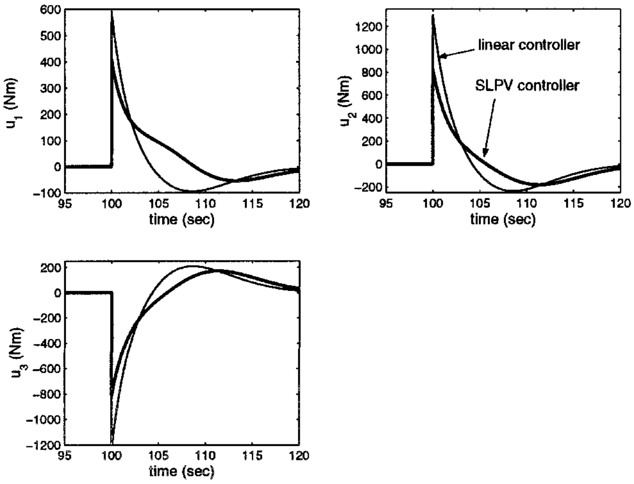

Fig. 2 Response of control torque $u_k, k = 1, 2, 3$.

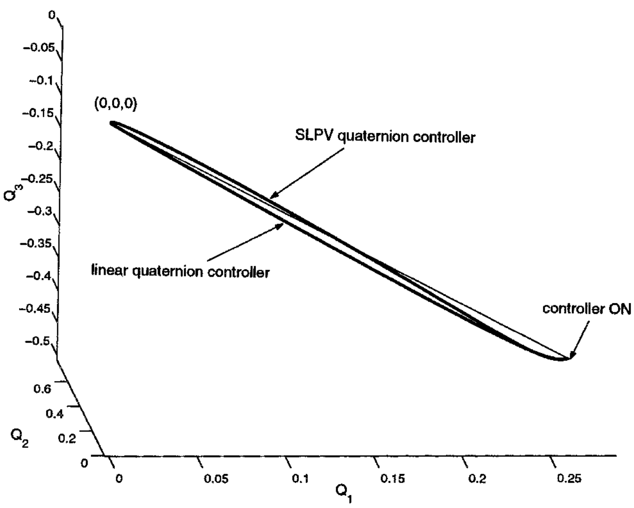

Fig. 3 Response of quaternion $Q_k, k = 1, 2, 3$.

the required amount of control energy

$$\int \|\boldsymbol{u}(t)\|^2 \, \mathrm{d}t$$

by 13.9%. These results imply that the SLPV controller leads to more fuel-efficient maneuvers.

Figure 3 shows the trajectories of the quaternion elements and a fictitious line to illustrate the exact eigenaxis maneuver. It is shown that, whereas the linear controller yields a trajectory with a kind of bias error, the SLPV controller leads to a trajectory with near-sinusoidal error that is closer to the exact eigenaxis rotation. These improvements are due to the new quaternion cross-coupling terms of the SLPV controller, which has also been observed by Lawton and Beard.[2]

V. Conclusions

A new technique is presented for designing quaternion feedback controllers to facilitate fuel-efficient eigenaxis maneuvers. The technique is based on the switched linear parameter-varying dissipation framework. The derived controller is shown to meet a challenging objective.

References

[1]Wie, B., Bailey, D., and Heiberg, C., "Rapid Multi-Target Acquisition and Pointing Control of Agile Spacecraft," *Proceedings of the 2000 AIAA Guidance, Navigation, and Control Conference* [CD-ROM], AIAA, Reston, VA, 2000.

[2]Lawton, J., and Beard, R., "Model Independent Eigenaxis Maneuvers Using Quaternion Feedback," *Proceedings of American Control Conference*, Inst. of Electrical and Electronics Engineers, New York, 2001, pp. 2339–2344.

[3]Lim, S., and How, J., "Analysis of Linear Parameter-Varying Systems Using a Nonsmooth Dissipative Systems Framework," *International Journal of Robust and Nonlinear Control*, Vol. 12, No. 12, 2002, pp. 1067–1092.

[4]Lim, S., and Chan, K., "Stability Analysis of Hybrid Linear Parameter-Varying Systems," *Proceedings of American Control Conference*, Inst. of Electrical and Electronics Engineers, New York, 2003.

[5]Lim, S., "New Quaternion Feedback Control for Efficient Large Angle Maneuvers," *Proceedings of the 2001 AIAA Guidance, Navigation, and Control Conference* [CD-ROM], AIAA, Reston, VA, 2001.

[6]Gahinet, P., Nemirovski, A., Laub, A., and Chilali, M., *The LMI Control Toolbox*, MathWorks, Inc., Natick, MA, 1995, pp. 9.27–9.30.